ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2025, Vol. 46, No. 3, pp. 1257–1265. © Pleiades Publishing, Ltd., 2025.

Linear Homogeneous Inequalities and Trajectory Routes of the Degenerate Lotka–Volterra Operators

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Received December 3, 2024; revised December 27, 2024; accepted January 6, 2025

Abstract—The work is dedicated to the study and finding of trajectory routes of degenerate Lotka–Volterra mappings. It is known that Lotka–Volterra mappings are automorphisms, which allows for the construction of both positive and negative orbits through the construction of a Lyapunov function and the analysis of the Jacobian matrix spectrum is accomplished in this work. The paper introduces a new definition of trajectory routes, the concept of positive and negative basins for stationary points, as well as velocities for mappings of this kind. Additionally, the proposed article presents a new approach to studying and finding routes by partitioning the simplex into parts according to the constructed signature. The analytical results obtained in this work are applicable in the fields of epidemiology, ecology, and economics.

2010 Mathematics Subject Classification: 54H20, 2010: 37B25, 37C25, 37C27

DOI: 10.1134/S1995080225605272

Keywords and phrases: *α and ω sets of trajectories, signature, trajectory speed, partitioning of the simplex, convex polyhedra, directed bigraphs, permissible trajectory routes*

INTRODUCTION

It is no secret to the mathematical community that one of the first mathematical models of interacting populations is the system of ordinary differential equations proposed by Vito Volterra, which historically arose in connection with attempts to explain the fluctuations in fish catches in the Adriatic Sea [1]. This same system, independently of Volterra, was proposed by Lotka earlier [2]. Based on these studies, research on these systems and their applications in continuous form continues to this day [3–8].

The discrete variant of the Lotka–Volterra operator on the simplex

$$S^{m-1} = \left\{ x \in \mathbb{R}^m : \sum_{i=1}^{m} x_i = 1;\ x_i \geq 0 \right\}$$

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is defined by the assignment of a real skew-symmetric matrix $A = (a_{ki})$ with the condition $|a_{ki}| \leq 1$ and acts according to the following law

$$x'_k = x_k \left(1 + \sum_{i=1}^{m} a_{ki} x_i\right), \quad k = \overline{1, m}, \tag{1}$$

introduced in works [9–12], where $x' = (x'_1, ..., x'_m) = Vx$ and $V : S^{m-1} \to S^{m-1}$.

It is clear that the solutions of the inequalities $\sum_{i=1}^{m} a_{ki} x_i > 0$ for some k, and $\sum_{i=1}^{m} a_{ki} x_i < 0$ for the remaining on the simplex S^{m-1} determine which coordinates increase and which decrease. It is evident that the sets of solutions to the inequalities, if they are non-empty, form an open convex polyhedron on the simplex. The trajectory routes are defined as the order of passage of the trajectory through these polyhedra. The work defines the possible trajectory routes of the operator V, when A is a block skew-symmetric matrix.

1. PRELIMINARY INFORMATION. STATEMENT OF THE PROBLEM

It is known [9, 10] that $V : S^{m-1} \to S^{m-1}$ under any $|a_{ki}| \leq 1$ is a homeomorphism, hence, in addition to the positive trajectory $\{x^{(n)}\}$, $n \in \mathbb{N}$, where $x^{(n+1)} = Vx^{(n)}$, $x^{(0)} \in S^{m-1}$, negative trajectories can also be considered, defined by the mapping $V^{-1} : S^{m-1} \to S^{m-1}$.

Let there be $\omega(x^{(0)})$ and $\alpha(x^{(0)})$ sets of limit points of the positive, respectively, negative trajectories starting from the point $x^{(0)} \in S^{m-1}$. As usual, $\mathrm{Fix}(V) = \{x \in S^{m-1} : Vx = x\}$, it is obvious that $\mathrm{extr}S^{m-1} \subset \mathrm{Fix}(V)$, where $\mathrm{extr}S^{m-1}$ is the set of extreme points of the simplex, i.e., the vertices of the simplex. If $x \in \mathrm{Fix}(V)$, then $\lambda = 1$ is always an eigenvalue of the Jacobian $J(V)$ at point x, and if x is an isolated fixed point, then the line containing the eigenvector corresponding to $\lambda = 1$ intersects the simplex S^{m-1} only at the point x.

In [9], it is proved that $P = \{x \in S^{m-1} : Ax \geq 0\} \neq \varnothing$, where any point $x \in P$ defines the Lyapunov function for the dynamical system (1).

Indeed, if $p = (p_1, ..., p_m) \in P$, then for function

$$\varphi(x) = \prod_{k=1}^{m} x_k^{p_k},$$

applying Jung's inequality [13], we get

$$\varphi_p(Vx) = \varphi_p(x) \prod_{k=1}^{m} \left(1 + \sum_{k=1}^{m} a_{ki} x_i\right)^{p_k} \leq \varphi_p(x) \left(1 - \sum_{k=1}^{m} \left(\sum_{k=1}^{m} a_{ik} p_k\right)\right) \leq \varphi_p(x), \tag{2}$$

where it is used that $a_{ki} = -a_{ik}$. Therefore, any negative trajectory converges, and $\alpha(x^{(0)}) \subset P$.

Let us introduce a new definition of the mappings speed.

Definition 1. *The vector $Vx - x$ is called the mappings speed V at a point x.*

Thus, velocity coordinates, which determine x_k, are increasing (not decreasing).

Let $\sigma(+ - + + ... -)$ an arbitrary set of signatures and $x \in S^{m-1}$ such that $Vx - x$ has the indicated signatures. It is obvious that the set of these x is a solution to the homogeneous inequalities (possibly empty). Let F_σ be a closure of a certain set. It is clear that F_σ is a convex polyhedron corresponding to the signature σ. For example, if $V : S^3 \to S^3$ has the form

$$V : \begin{cases} x'_1 = x_1(1 + x_2 - x_3 + x_4), \\ x'_2 = x_2(1 - x_1 + x_3 - x_4), \\ x'_3 = x_3(1 + x_1 - x_2 + x_4), \\ x'_4 = x_4(1 - x_1 + x_2 - x_3), \end{cases} \tag{3}$$

then the signature $\sigma(+ - +-)$ defines a polyhedron with vertices

$$e_1(1,0,0,0), \quad e_4(0,0,0,1), \quad M_1\left(\frac{1}{2},\frac{1}{2},0,0\right), \quad M_2\left(0,0,\frac{1}{2},\frac{1}{2}\right),$$

that is $F_\sigma = \mathrm{co}\,(e_1, e_4, M_1, M_2)$.

In general, if $F_\sigma \neq \varnothing$, then $\dim F_\sigma = m-1$ according to the principle of the non-existence of equality. Note that $\cup_\sigma F_\sigma = S^{m-1}$. So, we obtain the partition of the simplex into convex polyhedra. At any boundary point F_σ, for at least one index, we have $x'_k = x_k$.

Definition 2. *The sequence of traversing the trajectory through polyhedra is called the trajectory route.*

It is clear [13, 14] that a skew-symmetric matrix determines a directed graph with m vertices corresponding to the signs of its elements a_{ki}.

Let us recall some concepts from graph theory [15–17]:

If the vertices of a graph are divided into two non-overlapping parts I and II, such that only vertices from different parts can be connected by an oriented edge, then the graph is called an oriented bipartite graph (bigraph). A bigraph is called complete, if any two vertices from different parts are connected by an edge. Evidently, if the skew-symmetric matrix A defines an oriented bigraph, then it can be brought to a block form. In this work, the conditions for the convergence of trajectories, the definition of α and α sets, and the routes of trajectories of the Lotka–Volterra mappings acting on S^4, and corresponding to the oriented bigraph are studied.

The discrete Lotka–Volterra mappings and their connections with elements of graph theory were considered in works [9, 12, 14, 18, 19], in particular, tournaments [9, 12, 14] and partially-oriented graphs [18, 19]. Applications of such mappings in epidemiology and ecology were discussed in works [18, 20–22]. The proposed article is dedicated to the finding, construction, and study of negative and positive trajectory routes of the degenerate Lotka–Volterra operator, which is an explicit discrete analog of continuous compartmental models [23–28]. However, the methods and approaches, i.e., the connection with bigraphs, the research using signatures in the simplex stratification, and the results of discrete Lotka–Volterra mappings proposed in this work are new.

2. LOTKA–VOLTERRA MAPPING TRAJECTORY ROUTES

In this part of the paper, we will consider one of the most interesting representatives of the Lotka–Volterra mapping, which preserves the four-dimensional simplex $V: S^4 \to S^4$. Let the skew-symmetric matrix have the form

$$A = \begin{pmatrix} 0 & 0 & a & -b & c \\ 0 & 0 & -d & e & -f \\ -a & d & 0 & 0 & 0 \\ b & -e & 0 & 0 & 0 \\ -c & f & 0 & 0 & 0 \end{pmatrix}, \tag{4}$$

where $0 < a, b, c, d, e, f \leq 1$. Then, the corresponding mapping $V: S^4 \to S^4$ has the form

$$V: \begin{cases} x'_1 = x_1(1 + ax_3 - bx_4 + cx_5), \\ x'_2 = x_2(1 - dx_3 + ex_4 - fx_5), \\ x'_3 = x_3(1 - ax_1 + dx_2), \\ x'_4 = x_4(1 + bx_1 - ex_2), \\ x'_5 = x_5(1 - cx_1 + fx_2). \end{cases} \tag{5}$$

Next, we introduce the following notation: vertices of the simplex S^4 through $e_1, ..., e_5$, $\Gamma_{12} = \mathrm{co}\{e_1, e_2\}$ is edge, $\Gamma_{345} = \mathrm{co}\{e_3, e_4, e_5\}$ is a face of the simplex, as well as

$$\delta_1 = ce - bf, \quad \delta_2 = cd - af, \quad \delta_3 = bd - ae, \quad \Delta = \delta_1 + \delta_2 + \delta_3. \tag{6}$$

Consider the case when $\delta_1, \delta_2, \delta_3 > 0$.

Note that $Fix(V) = \Gamma_{12} \cup \Gamma_{345}$. Taking into account the positivity of the expressions δ_1, δ_2, and δ_3, we mark the points on the edges of their disposition

$$M_1 = \left(\frac{d}{a+d}, \frac{a}{a+d}, 0, 0, 0\right), \quad M_2 = \left(\frac{e}{b+e}, \frac{b}{b+e}, 0, 0, 0\right), \quad M_3 = \left(\frac{f}{c+f}, \frac{c}{c+f}, 0, 0, 0\right),$$

$$M_4 = \left(0, 0, \frac{b}{a+b}, \frac{a}{a+b}, 0\right), \quad M_5 = \left(0, 0, \frac{d}{d+e}, \frac{e}{d+e}, 0\right), \quad M_6 = \left(0, 0, 0, \frac{c}{b+c}, \frac{b}{b+c}\right),$$

$$M_7 = \left(0, 0, 0, \frac{f}{e+f}, \frac{e}{e+f}\right), \quad M_8 = \left(0, 0, \frac{\delta_1}{\Delta}, \frac{\delta_2}{\Delta}, \frac{\delta_3}{\Delta}\right),$$

with the help of which we determine the increase or decrease of a given coordinate under the action of the operator V. Since $\delta_1, \delta_2, \delta_3 > 0$ we obtain a picture of the partition of the faces Γ_{12} and Γ_{345}, shown in Fig. 1.

Considering $a\delta_1 - b\delta_2 + c\delta_3 = 0$ and $-d\delta_1 + e\delta_2 - f\delta_3 = 0$, we obtain solutions to the inequalities

$$P = \left\{x \in S^4 : Ax \geq 0\right\} = \text{co}\left\{M_6, M_7, M_8\right\}, \quad Q = \left\{x \in S^4 : Ax \leq 0\right\} = \text{co}\left\{M_4, M_5, M_8\right\}.$$

Theorem 1. *Under the condition $\delta_1, \delta_2, \delta_3 > 0$, any trajectory converges, and if $x^{(0)} \in riS^4$, i.e., $x^{(0)}$ is an interior point of a simplex. Then, we have $\alpha(x^{(0)}) \subset P$, $\omega(x^{(0)}) \subset Q$.*

Proof. Note that $\text{ri}S^4 \cap \text{Fix}(V) = \varnothing$. Let $x \in \text{ri}S^4$ and $p \in \text{ri}P$. Then, $p = (0, 0, p_3, p_4, p_5)$, where

$$p_3 = \frac{\delta_1\lambda_3}{\Delta}, \quad p_4 = \frac{c\lambda_1}{b+c} + \frac{f\lambda_2}{e+f} + \frac{\delta_2\lambda_3}{\Delta}, \quad p_5 = \frac{b\lambda_1}{b+c} + \frac{e\lambda_2}{e+f} + \frac{\delta_3\lambda_3}{\Delta},$$

and also $\lambda_1, \lambda_2, \lambda_3 > 0$ and $\lambda_1 + \lambda_2 + \lambda_3 = 1$. Consider the function on S^4

$$\varphi_p(x) = x_3^{p_3} x_4^{p_4} x_5^{p_5}, \tag{7}$$

which reaches its maximum only at the point $x = p$. Using Young's [13] inequalities, we obtain

$$\varphi_p(Vx) = \varphi_p(x)\,(1 - ax_1 + dx_2)^{p_3}\,(1 + bx_1 - ex_2)^{p_4}\,(1 - cx_1 + fx_2)^{p_5}$$
$$\leq \varphi_p(x)\,[1 - (ap_3 - bp_4 + cp_5)\,x_1 - (-dp_3 + ep_4 - fp_5)\,x_2].$$

Substituting the values p_3, p_4, and p_5, we find

$$ap_3 - bp_4 + cp_5 = \frac{\delta_1\lambda_2}{e+f}, \quad -dp_3 + ep_4 - fp_5 = \frac{\delta_1\lambda_1}{b+c}.$$

Consequently,

$$\varphi_p(Vx) \leq \varphi_p(x)\left(1 - \frac{\delta_1\lambda_2}{e+f}x_1 - \frac{\delta_1\lambda_1}{b+c}x_2\right) < \varphi_p(x), \tag{8}$$

as $\lambda_1, \lambda_2 > 0$. Since $V : S^4 \to S^4$ is a homeomorphism, then (7) can be rewritten as

$$\varphi_p(V^{-1}x) > \varphi_p(x) \tag{9}$$

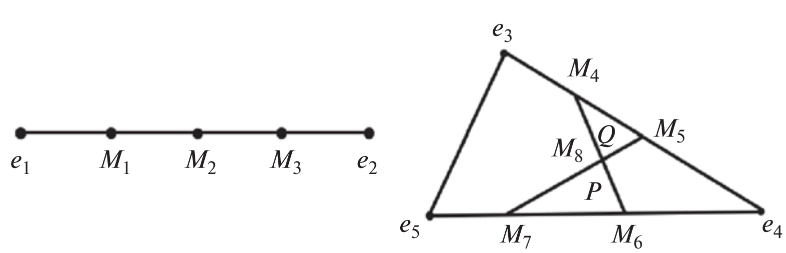

Fig. 1. The partition of the faces of Γ_{12} and Γ_{345} of the S^4 simplex.

for any $x \in riS^4$. □

Consequently, the sets $\{x \in S^4 : \varphi_p(x) \geq t\}$, where $0 < t \leq \varphi_p(p)$ are invariant under the mapping V^{-1}, and the sequence $\varphi_p\left(x^{(-n)}\right)$, $n = 1, 2, ...$ increases monotonically for any $x^{(0)} \in riS^4$. Therefore, any negative trajectory converges and $\alpha(x^{(0)}) \subset riP$.

Having calculated the Jacobian of the mapping $V : S^4 \to S^4$ at each fixed point we obtain:

1) eigenvalues are real and non-negative;

2) at all points of the edge Γ_{12}, with the exception of M_1, M_2, and M_3, the eigenvalue equal to $\lambda = 1$ has a multiplicity equal to two. At points M_1, M_2, and M_3 the multiplicity of the eigenvalue $\lambda = 1$ is three;

3) all fixed points of the face Γ_{345} with the exception of fixed points M_4, M_5, M_6, M_7, M_8 have an eigenvalue $\lambda = 1$ of multiplicity equal to three, and at points M_4, M_5, M_6, and M_7 the eigenvalue equal to $\lambda = 1$ has a multiplicity of four, and at the point M_8 the multiplicity of this eigenvalue is five;

4) all fixed points from riP are repellers;

5) fixed points from riQ are attractors.

Based on the analysis of the spectrum of the Jacobian, we obtain the picture, showing the Fig. 2 for the eigenvalues $J(V)$.

Inside the polyhedron co$\{e_1, e_3, e_4, e_5, M_1\}$ the coordinate $x_3^{(n)}$ decreases, $x_4^{(n)}$ increases, $x_5^{(n)}$ decreases, i.e., at fixed points from ri co$\{e_1, M_1\}$ two eigenvalues are less than 1, and one eigenvalue is greater than 1. Therefore, positive trajectories converge and $\omega(x^{(0)}) \subset riQ$.

Definition 3. *Let $x^* \in FixV$. The set $B_+(x^*) = \{x^{(0)} \in S^4 : x^{(n)} \to x^*\}$ is called the positive basin of the fixed point x^*.*

The negative basin $B_-(x^*)$ is defined similarly. A fixed point is neutral with respect to the mapping V, if $B_+(x^*) = x^*$.

Accordingly, $x^* \in \mathrm{Fix}(V)$ is neutral with respect to V^{-1} if $B_-(x^*) = x^*$. If $x^*, y^* \in \mathrm{Fix}(V)$ and $x^* \neq y^*$, then $B_+(x^*) \cap B_+(y^*) = \varnothing$. By the Grobman–Hartman theorem [29], if $x^* \in \mathrm{ri}Q$, then $\mathrm{dim}B_+(x^*) = 2$ and $B_-(x^*) = x^*$.

Remark. Let $V_\varepsilon : S^4 \to S^4$ be defined by the equalities

$$x'_k = x_k\left(1 + \varepsilon\sum_{i=1}^{5} a_{ki}x_i\right), \quad k = \overline{1,5}, \tag{10}$$

where $0 < \varepsilon \leq 1$.

Obviously, V_ε and V are topologically equivalent, since $V(\varepsilon x) = \varepsilon V_\varepsilon(x)$ for any $x \in \mathbb{R}^5$. In this case, the fixed points, the basins of fixed points of the operators V and V_ε coincide. Since $||V_\varepsilon x - x||$ can be made arbitrarily small by choosing $\varepsilon > 0$, then it is permissible to assume that $V_\varepsilon x - x$ and $V_\varepsilon^2 x - V_\varepsilon x$ have the same signs in all coordinates.

According to Fig. 2, we have 16 possible signatures:

$$\sigma_1 = (+--+-), \quad \sigma_2 = (+-++-), \quad \sigma_3 = (+-+--), \quad \sigma_4 = (+-+-+),$$

$$\sigma_5 = (++-+-), \quad \sigma_6 = (++++-), \quad \sigma_7 = (+++--), \sigma_8 = (+++-+),$$

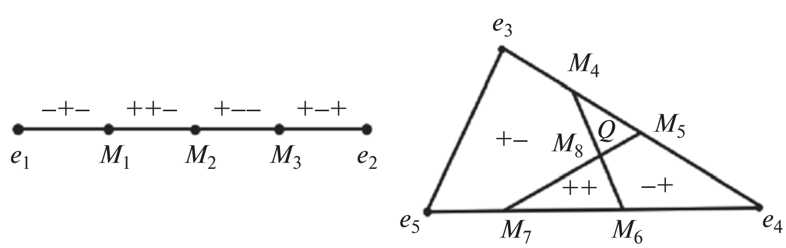

Fig. 2. Analysis of the Jacobian spectrum of the $J(V)$.

$$\sigma_9=(-+-+-),\quad \sigma_{10}=(-+++-),\quad \sigma_{11}=(-++--),\quad \sigma_{12}=(-++-+),$$

$$\sigma_{13}=(---+-),\quad \sigma_{14}=(--++-),\quad \sigma_{15}=(--+--),\quad \sigma_{16}=(--+-+).$$

The following polyhedras correspond to these signatures:

$$F_1=\mathrm{co}\left\{e_1,e_3,e_5,M_1,M_4,M_7,M_8\right\},\quad F_2=\mathrm{co}\left\{e_3,e_5,M_1,M_2,M_4,M_7,M_8\right\},$$

$$F_3=\mathrm{co}\left\{e_3,e_5,M_2,M_3,M_4,M_7,M_8\right\},\quad F_4=\mathrm{co}\left\{e_2,e_3,e_5,M_4,M_7,M_8\right\},$$

$$F_5=\mathrm{co}\left\{e_1,M_1,M_6,M_7,M_8\right\},\quad F_6=\mathrm{co}\left\{M_1,M_2,M_6,M_7,M_8\right\},$$

$$F_7=\mathrm{co}\left\{M_2,M_3,M_6,M_7,M_8\right\},\quad F_8=\mathrm{co}\left\{e_2,M_3,M_6,M_7,M_8\right\},$$

$$F_9=\mathrm{co}\left\{e_1,e_4,M_1,M_5,M_6,M_8\right\},\quad F_{10}=\mathrm{co}\left\{e_4,M_1,M_2,M_5,M_6,M_8\right\},$$

$$F_{11}=\mathrm{co}\left\{e_4,M_2,M_3,M_5,M_6,M_8\right\},\quad F_{12}=\mathrm{co}\left\{e_2,e_4,M_3,M_5,M_6,M_8\right\},$$

$$F_{13}=\mathrm{co}\left\{e_1,M_1,M_4,M_5,M_8\right\},\quad F_{14}=\mathrm{co}\left\{M_1,M_2,M_4,M_5,M_8\right\},$$

$$F_{15}=\mathrm{co}\left\{M_2,M_3,M_4,M_5,M_8\right\},\quad F_{16}=\mathrm{co}\left\{e_2,M_3,M_4,M_5,M_8\right\}.$$

It is obvious in this case $\bigcap_{k=1}^{16}F_k=M_8$. Prove a more general statement about the placement of polyhedra corresponding to admissible signatures.

Lemma 1. *The intersection of any two polyhedra is either empty or is a common face.*

Proof. On any face $x'_k=x_k$ for at least one index k. It is clear that the equality $x'_k=x_k$ is preserved on the affine hull of this face. Consequently, the affine hull of this face cannot contain an interior point of any other polyhedron, since in the interior points of any polyhedron $x'_k\neq x_k$ for all indices k. Therefore, the intersection of any two polyhedra must be a common face or empty. Remember that $\dim F_k=4$ for all $k=\overline{1.16}$. □

Definition 4. *Two polyhedra are called adjacent if they have a common face of dimension 3.*

In the case under consideration, we obtain the following adjacency graph for the polyhedra $F_1,\ldots,F_{16}$ (see Fig. 3).

In Fig. 3, the first and last lines coincide, therefore, identifying them, we obtain that the adjacency graph of polyhedra is a cylindrical graph. Recall that a cylindrical graph is a graph that has at least one closed contour [15–17].

Let be F_i and F_j are adjacent polyhedra, T their common face. If $||Vx-x||$ is small enough, due to the choice of $\varepsilon>0$, then $V(T)\subset F_i\cup F_j$. Since $\dim T=3$, then $T\cap\Gamma_{12}\neq\varnothing$ and $T\cap\Gamma_{345}\neq\varnothing$, hence, $V\left(T\cap\Gamma_{12}\right)=T\cap\Gamma_{12}$, $V\left(T\cap\Gamma_{345}\right)=T\cap\Gamma_{345}$.

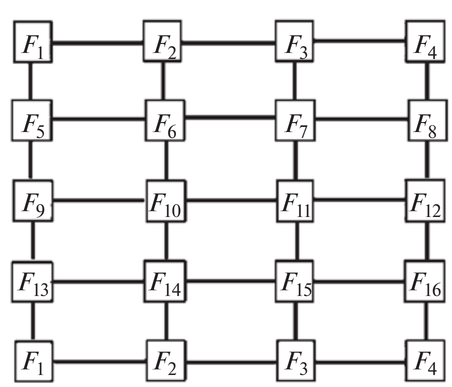

Fig. 3. The adjacency graph for the polyhedra $F_1,\ldots,F_{16}$.

If $V(T) \subset F_i$, then the route follows from F_j to F_i. We mark this case as $F_j \to F_i$. The notation $F_i \leftrightarrow F_j$ means the existence of routes going both from F_i to F_j and back. Since V is a homeomorphism, $V(T)$ is a closed set.

Let $T = F_1 \cap F_2$. Knowing $\sigma_1(+--+-)$, $\sigma_2(+-++-)$, we obtain that $x_3' = x_3$ for any $x \in T$ and

$$x_1' > x_1, \quad x_2' < x_2, \quad x_4' > x_4, \quad x_5' < x_5$$

for all $x \in riT$. In this case,

$$x_3'' = x_3'' \left(1 - ax_1' + dx_2'\right) < x_3 \left(1 - ax_1 + dx_2\right) = x_3'.$$

Therefore, $x_3'' < x_3'$, i.e., $V(T) \subset F_1$. Thus, the route of some trajectories may follow from F_2 to F_1.

If $x \in F_1 \cap F_5$, then taking into account the signatures $\sigma_1 = (+--+-)$, $\sigma_2 = (++-+-)$ we find that

$$x_2' = x_2, \quad x_1' > x_1, \quad x_3' < x_3, \quad x_4' > x_4, \quad x_5' < x_5.$$

Therefore,

$$x_2'' = x_2' \left(1 - dx_3' + ex_4' - fx_5'\right) > x_2 \left(1 - dx_3 + ex_4 - fx_5\right) = x_2'.$$

Hence, $V(F_1 \cap F_5) \subset F_5$ and we have the transition $F_1 \to F_5$.

Consider adjacent polyhedra F_5 and F_6. If $x \in F_5 \cap F_6$, then, according to $\sigma_5 = (++-+-)$ and $\sigma_6 = (++++-)$, we have $x_3' = x_3$ and inequalities

$$x_1^{'} > x_1, \quad x_2^{'} > x_2, \quad x_4^{'} > x_4, \quad x_5^{'} < x_5.$$

Then,

$$x_3'' = x_3' \left(1 - ax_1' + dx_2'\right). \tag{11}$$

Since $F_5 \cap F_6 = \mathrm{co}\{M_1, M_6, M_7, M_8\}$, then $x \in F_5 \cap F_6$ can be uniquely represented in the form

$$x = \left(\frac{d\lambda_1}{a+d}, \frac{a\lambda_1}{a+d}, \frac{\delta_1\lambda_4}{\Delta}, \frac{c\lambda_2}{b+c} + \frac{f\lambda_3}{e+f} + \frac{\delta_2\lambda_4}{\Delta}, \frac{b\lambda_2}{b+c} + \frac{e\lambda_3}{e+f} + \frac{\delta_3\lambda_4}{\Delta}\right),$$

where $\lambda_1, ..., \lambda_4 \geq 0$ and $\sum_{i=1}^4 \lambda_i = 1$. In respect that $a\frac{\delta_1}{\Delta} - b\frac{\delta_2}{\Delta} + c\frac{\delta_3}{\Delta} = -\frac{d\delta_1}{\Delta} + \frac{e\delta_2}{\Delta} - \frac{f\delta_3}{\Delta} = 0$, we have

$$x_1' = \frac{d\lambda_1}{a+d}\left(1 + \frac{\delta_1\lambda_3}{e+f}\right), \quad x_2' = \frac{a\lambda_1}{a+d}\left(1 + \frac{\delta_1\lambda_2}{b+c}\right). \tag{12}$$

Substituting these values into (10), we find

$$x_3'' = x_3'\left(1 - \frac{ad\lambda_1}{a+d}\left(1 + \frac{\delta_1\lambda_3}{e+f}\right) + \frac{ad\lambda_1}{a+d}\left(1 + \frac{\delta_1\lambda_2}{b+c}\right)\right) = x_3'\left(1 + \delta_1\left(\frac{\lambda_2}{b+c} - \frac{\lambda_3}{e+f}\right)\right).$$

So, for $\frac{\lambda_2}{b+c} > \frac{\lambda_3}{e+f}$ we have $x_3'' > x_3'$ and $x_3'' < x_3'$ at $\frac{\lambda_2}{b+c} < \frac{\lambda_3}{e+f}$ (see Fig. 4).

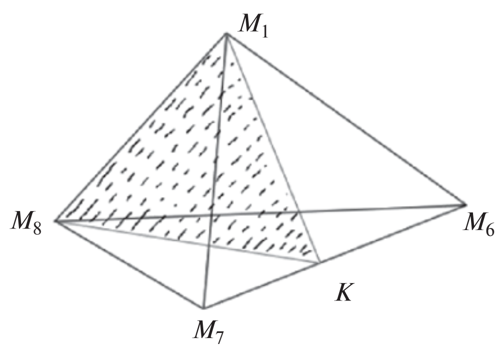

Fig. 4. The section of the simplex, according to the signatures.

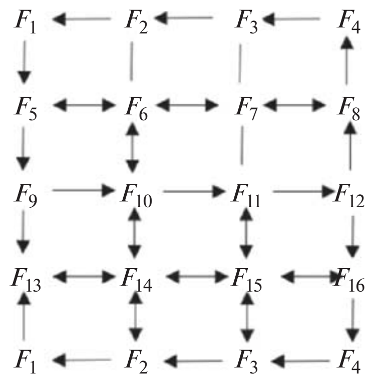

Fig. 5. The route of the trajectories.

Thus, the plane M_1M_8K divides the points $F_5 \cap F_6$ into those passing from F_5 to F_6 and back from F_6 to F_5.

Theorem 2. *Having determined the order of the trajectory in all adjacent polyhedra, we obtain oriented graph shown in Figure 5.*

Similarly, outside $F_5 \cup F_6 \cup F_7 \cup F_8$ there is only a finite part of the negative trajectory.

3. CONCLUSIONS AND DISCUSSION

As we stated above, the paper proposes a study of a different class of Lotka–Volterra mappings with a new approach and research method. All possible signatures and the corresponding sections of the simplex are considered. And also for the first time it is shown that some of the discrete class of Lotka–Volterra mappings, unlike those introduced in [9, 12, 14], can be associated with bigraphs.

In this work, using the construction of Lyapunov functions and analysis of the spectrum of the Jacobian, positive and negative trajectories are constructed, and their limit sets are found (Theorem 1). New definitions for the concepts speed of mapping and trajectory route have been introduced. For the mapping under study, all possible signatures were considered and the corresponding polyhedra $F_1, ..., F_{16}$ were constructed and an adjacency graph for them was obtained. As a result, the order of the trajectory in all adjacent polyhedra is determined and an oriented graph is constructed (see Theorem 2 and Fig. 5). The considered case of the Lotka–Volterra mapping is interesting in that it is a full-fledged representative of a discrete analogue of compartmental models of epidemiology and ecology. We will apply such a model to a specific situation in the following work.

FUNDING

This research is being conducted as part of the project no. BR 27100483 "Development of predictive exploration technologies for identifying ore-prospective areas based on data analysis from the unified subsurface user platform 'Minerals.gov.kz' using artificial intelligence and remote sensing methods."

CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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